

The virtual cohomology dimension of Teichmüller modular groups: the first results and a road not taken

*To Leningrad Branch of Steklov Mathematical Institute,
with gratitude for giving me freedom to
pursue my interests in 1976–1998.*

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Preface

Sections 1, 3, and 4 of this paper are based on my paper [Iv2]. Section 2 is devoted to the motivation behind the results of [Iv2]. Section 5 is devoted to the context of my further results about the virtual cohomological dimension of Teichmüller modular groups. The last Section 6 is devoted to the mathematical and non-mathematical circumstances which shaped the paper [Iv2] and some further developments.

As I only recently realized, [Iv2] contains the nucleus of some techniques for working with complexes of curves and other similar complexes used many times by myself and then by other mathematicians. On the other hand, one of the key ideas of [Iv2], namely, the idea of using the Hatcher–Thurston cell complex [6], was abandoned by me already in 1983, and was not taken up by other mathematicians. It seems that it still holds some promise. This is a *road not taken*,

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alluded to in the title. The implied reference to a famous poem by Robert Frost is indeed relevant, especially if the poem is understood not in the clichéd way.

The list of references consists of two parts. The first part reproduces the list of references of [Iv2]. The second part consists of additional references. The papers from the first list are referred to by numbers, and from the second one by letters followed by numbers. So, [2] refers to the first list, and [Iv2] to the second.

The exposition and English in Sections 1, 3, and 4 are, I hope, substantially better than in [Iv2]. At the same time, these sections closely follow [Iv2] with one exception. Namely, the original text of [Iv2] contained a gap. A densely written correction was added to [Iv2] at the last moment as an additional page. In the present paper this correction is incorporated into the proof of Lemma 13 (see the subsection 15, Claims 1 and 2). Finally, \LaTeX leads to much better output than a typewriter combined with writing in formulas by hand.

1. Introduction

Let X_g be a closed orientable surface of genus $g \geq 2$. The *Teichmüller modular group* Mod_g of genus g is defined as the group of isotopy classes of diffeomorphisms $X_g \rightarrow X_g$, i.e. $\text{Mod}_g = \pi_0(\text{Diff}(X_g))$. The group may be also defined as the group of homotopy classes of homotopy equivalences $X_g \rightarrow X_g$. These two definitions are equivalent by a classical result of Baer–Nielsen. See [8] for a modern exposition close in spirit to the original papers of J. Nielsen and R. Baer. It is well known that X_g is a $K(\pi, 1)$ -space, where $\pi = \pi_1(X_g)$. This allows to determine the group of homotopy classes of homotopy equivalences $X_g \rightarrow X_g$ in terms of the fundamental group $\pi_1(X_g)$ alone and conclude that Mod_g is isomorphic to the outer automorphisms group

$$\text{Out}(\pi_1(X_g)) = \text{Aut}(\pi_1(X_g)) / \text{Inn}(\pi_1(X_g)).$$

Here $\text{Aut}(\pi)$ denotes the automorphism group of a group π and $\text{Inn}(\pi)$ denotes the subgroup of inner automorphisms of π . The groups Mod_g are also known as the *surface mapping class groups*.

The present paper is devoted to what was the first step toward the computation of the virtual cohomology dimension of the groups Mod_g . Its main result (Theorem 1 below) provided the first non-trivial estimate of the virtual cohomology dimension of Mod_g .

The ordinary cohomology dimension of Mod_g is infinite because Mod_g contains non-trivial elements of finite order. However Mod_g is virtually torsion free, i.e. Mod_g contains torsion free subgroups of finite index (this result is due to J.-P. Serre; see [3]). If a group Γ is virtually torsion free, then all torsion free subgroups of Γ of finite index

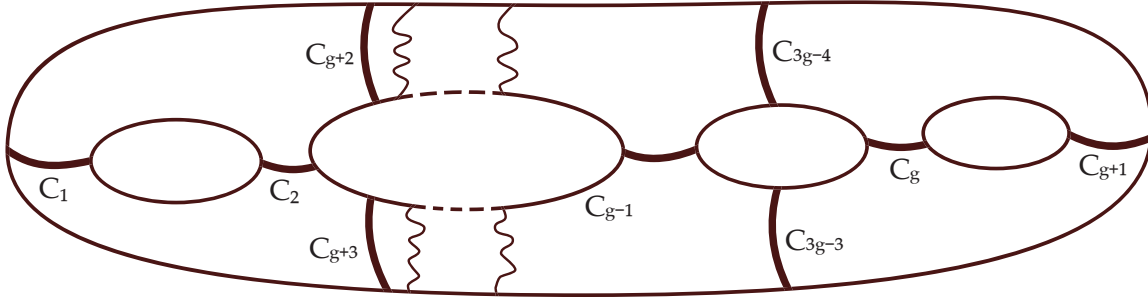


Figure 1: A maximal collection of disjoint pair-wise non-isotopic circles.

have the same cohomology dimension, which is called the *virtual cohomology dimension* of Γ and is denoted by $\text{vcd}(\Gamma)$. It is well known that $\text{vcd}(\text{Mod}_g)$ is finite and moreover that $\text{vcd}(\text{Mod}_g) = 1$ for $g = 1$ and $3g - 3 \leq \text{vcd}(\text{Mod}_g) \leq 6g - 7$ for $g \geq 2$.

Let me recall the proofs of the last inequalities (assuming that $g \geq 2$). In order to prove the first one, recall that Mod_g contains a free abelian subgroup of rank $= 3g - 3$. For example, the subgroup generated by Dehn twists along the curves C_1, \dots, C_{3g-3} on Fig. 1 is a free abelian subgroup of rank $= 3g - 3$. Since the virtual cohomology dimension cannot be increased by passing to a subgroup, and since $\text{vcd}(\mathbb{Z}^n) = n$, we see that $3g - 3 \leq \text{vcd}(\text{Mod}_g)$.

The proof of the inequality $\text{vcd}(\text{Mod}_g) \leq 6g - 7$ is much more deep and is based on theories of Riemann surfaces and of Teichmüller spaces. Recall that Mod_g naturally acts on the Teichmüller space T_g of marked Riemann surfaces of genus g . The action of Mod_g on T_g is a properly discontinuous, and the quotient space T_g/Mod_g is the *moduli space* of Riemann surfaces of genus g (this is the source of the term *Teichmüller modular group*). Moreover, any torsion free subgroup Γ of Mod_g acts on T_g freely. Since T_g is homeomorphic to \mathbb{R}^{6g-6} , for such a subgroup Γ the quotient space T_g/Γ is a $K(\Gamma, 1)$ -space (and, in addition, is a manifold). This implies that the cohomology dimension of Γ is $\leq \dim T_g/\Gamma = 6g - 6$, and hence $\text{vcd}(\text{Mod}_g) \leq 6g - 6$. In order to prove that, moreover, $\text{vcd}(\text{Mod}_g) \leq 6g - 7$, recall that T_g/Mod_g is non-compact, and hence T_g/Γ is also non-compact. Since the n -th cohomology groups of n -dimensional non-compact manifolds with any coefficients, including twisted ones, are equal to 0, this implies that the $6g - 6$ -th cohomology group of any such subgroup Γ is equal to 0, and hence the cohomology dimension of Γ is $< 6g - 6$. It follows that $\text{vcd}(\text{Mod}_g) < 6g - 6$, i.e. $\text{vcd}(\text{Mod}_g) \leq 6g - 7$.

The main result of the present paper is the following strengthening of the inequality $\text{vcd}(\text{Mod}_g) \leq 6g - 7$.

1. Theorem. $\text{vcd}(\text{Mod}_g) \leq 6g - 9$ for $g \geq 2$ and $\text{vcd}(\text{Mod}_2) = 3$. In addition, Mod_2 is

virtually a duality group of dimension 3.

The proof of this theorem is based on the properties of a boundary of Teichmüller space introduced by W. Harvey [4, 5]. The key property of Harvey's boundary of T_g is the fact that it is homotopy equivalent to (the geometric realization of) a simplicial complex $\mathcal{C}(X_g)$. The complex $\mathcal{C}(X_g)$ was also introduced by W. Harvey and is known as the *complex of curves* of X_g . We will recall the definition of complexes of curves in Section 3. Using the results of W. Harvey and the theory of cohomology of groups, especially the theory of groups with duality developed by R. Bieri and B. Eckman Theorem 1 can be reduced to the following theorem (see Section 3).

2. Theorem. *The complex of curves $\mathcal{C}(X_g)$ of X_g is simply-connected for $g \geq 2$.*

Using the same arguments one can deduce from Theorem 2 that Mod_2 is virtually a duality group in the Bieri-Eckmann sense [1], i.e. that Mod_2 contains a subgroup of finite index which is a duality group in the Bieri-Eckmann sense. Theorem 2 is deduced from the simply-connectedness of a cell complex introduced by A. Hatcher and W. Thurston [6] (see Section 4). The simply-connectedness of the Hatcher-Thurston complex is one of the main results of their paper [6].

The rest of the paper is arranged as follows. In Section 3 we review the basic properties of Harvey boundary of Teichmüller space, and then deduce Theorem 1 from Theorem 2. In Section 4 we start with defining complexes of curves and Hatcher-Thurston complexes, and then deduce Theorem 2 from results of A. Hatcher and W. Thurston [6].

In Section 2 we explain the ideas from the theory of arithmetic groups which served as a motivation for the approach to the virtual cohomology dimension of Mod_g outlined above, and for the further work in this direction. In Section 5 we outline a broad context in which Theorem 2 and then stronger results about the connectivity of $\mathcal{C}(X_g)$ were discovered. Section 6 is the last one and is devoted to some personal reminiscences related to these stronger results. It has grown out of a short summary written by me in Summer of 2007 as a step toward writing the expository part of the paper [Iv-J] by Lizhen Ji and myself.

2. Motivation from the theory of arithmetic groups

The Borel-Serre theory. Around 1970 A. Borel and J.-P. Serre studied cohomology of arithmetic [BS1] and S -arithmetic [BS2] groups. In particular, Borel and Serre computed the virtual cohomology dimension of such groups. The details were published in [2] and [BS3] respectively.

In outline, Borel and Serre approach is as follows. Let Γ be an arithmetic group. There is a natural contractible smooth manifold X on which Γ acts. Moreover, Γ acts on X properly discontinuously, and a subgroup of finite index in Γ acts on X freely. For the purposes of computing or estimating $\text{vcd}(\Gamma)$, we can replace Γ by such subgroup, if necessary, and assume that Γ itself acts on X freely. Then the quotient X/Γ is a $K(\Gamma, 1)$ -space and one may hope to use it for understanding the cohomological properties of Γ . Unfortunately, X/Γ is usually non-compact.

The first step of the Borel-Serre approach [BS1], [2] is a construction of a natural compactification of X/Γ . This compactification has the form \bar{X}/Γ , where \bar{X} is a *smooth manifold with corners* independent of Γ and having X as its interior. As a topological space, a *smooth manifold with corners* is a topological manifold with boundary. It has also a canonical structure similar to that of smooth manifold with boundary: while the smooth manifolds with boundary are modeled on products $\mathbf{R}^n \times \mathbf{R}_{\geq 0}$ (where $n+1$ is equal to the dimension), the smooth manifolds with corners are modeled on products $\mathbf{R}^n \times \mathbf{R}_{\geq 0}^m$ (where $n+m$ is the dimension).

The existence of a structure of smooth manifold with corners on \bar{X} together with the compactness of \bar{X} implies that \bar{X}/Γ admits a *finite* triangulation. In particular, \bar{X}/Γ is homotopy equivalent to a *finite* CW-complex. This implies that the virtual cohomological dimension $\text{vcd}(\Gamma)$ is *finite*. In fact, this implies a much stronger finiteness property of Γ . Namely, Γ is a *group of type (FL)*, i.e. there exists a resolution of the trivial Γ -module \mathbf{Z} by *finitely generated free modules* and having *finite length*.

The second step of the Borel-Serre method is an identification of the homotopy type of the boundary ∂X . Borel and Serre proved that ∂X is homotopy equivalent to the (geometric realization of the) Tits building associated with X (or, one may say, with Γ). By a theorem of L. Solomon and J. Tits (see [So], [Ga]), the Tits building is homotopy equivalent to a wedge of spheres. Moreover, all these spheres have the same dimension, equal to $r-1$, where r is the so-called *rank* of X (or of Γ).

The last step in the computation of $\text{vcd}(\Gamma)$ by Borel-Serre method is an application of a version of the Poincaré-Lefschetz duality (namely, of the version allowing arbitrary twisted coefficients). This step uses the fact that Γ is a group of type (FL) implied by the existence of a structure of a smooth manifold with corners on \bar{X} . In fact, it would be sufficient to know that Γ is a group of type (FP), i.e. there exists a resolution of the trivial Γ -module \mathbf{Z} by *finitely generated projective modules* and having *finite length*.

If Γ is only a S -arithmetic group, there is still a natural contractible smooth manifold X on which Γ acts. But in this case Γ does not act on X properly discontinuously. In order to overcome this difficulty Borel and Serre [BS2], [BS3] multiplied X by another topological space Y with a canonical action of Γ . The space Y is not a smooth or topological manifold. In fact, its topology is closely related to the topology of non-archimedean local fields. This is the source of the main difficulties in the case of S -arithmetic groups compared to the arithmetic ones. These difficulties are technically

irrelevant for Teichmüller modular groups. But the fact the original Borel-Serre theory can be applied in a situation different from the original one was encouraging.

The Bieri-Eckmann theory. While the Borel-Serre theory served as the motivation, on the technical level it is easier to use more general results of Bieri-Eckmann [1].

In fact, the last step of the Borel-Serre computation of $\text{vcd}(\Gamma)$ works in a very general situation. The corresponding general theory is due to R. Bieri and B. Eckmann [1], who developed it independently of Borel-Serre. Bieri and Eckmann [1] presented a polished theory ready for applications. In the detailed publication [BS3] of their results about S -arithmetic groups Borel and Serre used [1] when convenient. The following easy corollary of Theorem 6.2 of Bieri-Eckman summarizes the results needed.

3. Theorem. *Suppose that a discrete group Γ acts freely on a topological manifold \bar{X} of dimension n with boundary $\partial\bar{X}$. Suppose that \bar{X}/Γ is homotopy equivalent to a finite CW-complex. If for some natural number d the reduced integral homology groups $H_i(\partial\bar{X})$ are equal to 0 for $i \neq d$ and the group $H_d(\partial\bar{X})$ is torsion-free, then $\text{vcd}(\Gamma) = n - 1 - d$. If the manifold \bar{X}/Γ is orientable, then Γ is a duality group in the sense of [1] and $\text{cd}(\Gamma) = \text{vcd}(\Gamma) = n - 1 - d$.*

In Borel-Serre theory $\partial\bar{X}$ is homotopy equivalent to a bouquet of spheres of the same dimension by the Solomon-Tits theorem. It follows that $H_i(\partial\bar{X}) = 0$ if $i \neq d$, where d is the dimension of these spheres, and that $H_d(\partial\bar{X})$ is a free abelian group. In particular, it is torsion-free, and hence Theorem 3 applies. The next theorem is not proved by Bieri-Eckmann [1], but is very close to Theorem 3.

4. Theorem. *In the framework of Theorem 3, if c is a natural number such that the reduced homology groups $H_i(\partial\bar{X}) = 0$ for $i \leq c - 1$, then $\text{vcd}(\Gamma) \leq n - 1 - c$.*

Since in the present paper we prove only upper estimates of the virtual cohomology dimension, Theorem 4 is better suited for our goals.

3. The Harvey boundary of Teichmüller space

An analogue for Teichmüller modular groups Mod_g of Borel-Serre manifolds X is well known since the work of Teichmüller. It is nothing else but the Teichmüller spaces T_g . Teichmüller modular group Mod_g acts on T_g discontinuously, and a subgroup of finite index acts freely by the results of Serre [Se].

Motivated by Borel-Serre theory, W. Harvey constructed in [5] an analogue of manifolds \bar{X} . Namely, Harvey constructed topological manifolds \bar{T}_g (with boundary) such that

$$T_g = \bar{T}_g \setminus \partial\bar{T}_g.$$

In other words, T_g is the interior of \bar{T}_g . Both \bar{T}_g and $\partial\bar{T}_g$ are non-compact. The boundary $\partial\bar{T}_g$ is called the *Harvey boundary* of T_g . The canonical action of Mod_g on T_g extends to \bar{T}_g by the continuity. This extended action has the following properties:

- (i) *the action is properly discontinuous;*
- (ii) *a subgroup of finite index in Mod_g acts on \bar{T}_g freely;*
- (iii) *the quotient space \bar{T}_g/Mod_g is compact.*

In particular, \bar{T}_g/Mod_g is a compactification of the moduli space T_g/Mod_g . In fact, \bar{T}_g is not only a topological manifold; it has a structure of a smooth manifold with corners. This structure is not completely canonical (a subtle choice is involved in its construction; see [Iv5]). But any natural construction of such a structure leads to a Mod_g -invariant structure. Therefore, we may assume that it is Mod_g -invariant. Then for any subgroup Γ of Mod_g acting freely on \bar{T}_g the quotient \bar{T}_g/Γ is a smooth manifold with corners.

We refer to the paper [2] by A. Borel and J.-P. Serre for the definition and the basic properties of manifolds with corners. Since the theory of the Harvey boundary is to a big extent modeled on the theory of A. Borel and J.-P. Serre [2], this seems to be the most natural reference. In the present paper we will need only one result of the theory of manifolds with corners; see the proof of the next Lemma.

5. Lemma. *Suppose that Γ is a subgroup of Mod_g of finite index in Mod_g . If Γ acts freely on \bar{T}_g , then \bar{T}_g/Γ is finitely triangulable space of type $K(\Gamma, 1)$.*

Proof. Since every topological manifold with boundary is homotopy equivalent to its interior, \bar{T}_g/Γ is homotopy equivalent to T_g/Γ . Since T_g/Γ is a $K(\Gamma, 1)$ -space, as we mentioned in Section 1, \bar{T}_g/Γ is also a $K(\Gamma, 1)$ -space. In addition, \bar{T}_g/Γ is a smooth manifold with corners. It is known that the corners of a smooth manifolds with corners can be smoothed (see [2]). Therefore, \bar{T}_g/Γ is homeomorphic to a smooth manifold (without corners). Since Γ is a subgroup of finite index in Mod_g , and \bar{T}_g/Mod_g is compact, the quotient \bar{T}_g/Γ is also compact. As is well known, every compact smooth manifold is finitely triangulable. Therefore, \bar{T}_g/Γ is finitely triangulable. This completes the proof of the lemma. ■

By the property (ii) of the Harvey boundary there is a subgroup Γ of Mod_g which acts on \bar{T}_g freely. By Lemma 5, the quotient \bar{T}_g/Γ admits a finite triangulation. In particular, it is homotopy equivalent to a finite CW-complex. Therefore, the action of Γ on \bar{T}_g fits into the framework of Theorems 3 and 4 (with $\bar{X} = \bar{T}_g$).

6. Lemma. *If the reduced homology groups $H_i(\partial\bar{T}_g) = 0$ for $i = 0, 1, \dots, c-1$, then*

$$\text{vcd}(\text{Mod}_g) \leq \dim T_g - 1 - c = 6g - 7 - c.$$

This is a special case of Theorem 4. The case $c = 2$ is sufficient for the applications in this paper, and we will prove Lemma 6 only in this case. See Lemma 9 below. In fact, this proof works *mutatis mutandis* in the general situation of Theorem 4.

Recall that a group Γ is called a *group of type (FL)* if the trivial Γ -module \mathbb{Z} admits a resolution of finite length consisting of finitely generated free Γ -modules.

7. Lemma. *Under the assumptions of Lemma 5, Γ is a group of type (FL).*

Proof. The lemma follows from Lemma 5 together with Proposition 9 of [Se]. ■

8. Lemma. *Let k be a natural number. If Γ is a group of finite cohomology dimension, and if $H^n(\Gamma, M) = 0$ for $n > k$ and all free Γ -modules M , then $\text{cd}\Gamma \leq k$.*

Proof. This lemma is due to R. Bieri and B. Eckmann [1]. See [1], Proposition 2.1. ■

9. Lemma. *If the reduced homology groups $H_0(\partial\bar{T}_g) = H_1(\partial\bar{T}_g) = 0$, then*

$$\text{vcd}(\text{Mod}_g) \leq 6g - 9.$$

Proof. Let Γ be a subgroup of finite index of Mod_g . We may assume that the action of Γ on T_g is free. It is sufficient to prove that under such assumptions the cohomology dimension $\text{cd}(\Gamma) \leq 6g - 9$. We start with the following claim.

CLAIM 1. *$H^n(\Gamma, \mathbb{Z}[\Gamma]) = 0$ for $n > 6g - 9$, where $\mathbb{Z}[\Gamma]$ is the integer group ring of Γ together with its standard structure of a right Γ -module (given by the multiplication in Γ).*

Proof of the claim. Since \bar{T}_g/Γ is finitely triangulable $K(\Gamma, 1)$ -space and \bar{T}_g is its universal covering (because \bar{T}_g is homotopy equivalent to T_g and hence is a contractible space), we can apply the results of R. Bieri and B. Eckmann [1] (see [1], Subsection 6.4). Their results imply that

$$H^n(\Gamma, \mathbb{Z}[\Gamma]) = H_{d-n-1}(\partial\bar{T}_g, \mathbb{Z})$$

for all n , where $d = \dim T_g$. By the assumptions of the lemma, $H_k(\partial\bar{T}_g, \mathbb{Z}) = 0$ for $k < 2$. Therefore $H^n(\Gamma, \mathbb{Z}[\Gamma]) = 0$ for $d - n - 1 < 2$, i.e. for $n > d - 3$. Hence $H^n(\Gamma, \mathbb{Z}[\Gamma]) = 0$ for $n > d - 3 = \dim T_g - 3 = 6g - 6 - 3 = 6g - 9$. This proves Claim 1. □

Recall that the module $\mathbf{Z}[\Gamma]$ is a free Γ -module with one free generator. The next step is to extend the above claim to arbitrary free modules.

CLAIM 2. *In M is a free Γ -module, then $H^n(\Gamma, M) = 0$ for $n > 6g - 9$.*

Proof of the claim. Since any finitely generated free Γ -module is isomorphic to a finite sum of copies of $\mathbf{Z}[\Gamma]$, Claim 1 implies that $H^n(\Gamma, M) = 0$ for $n > 6g - 9$ for any finitely generated free module M . By Corollary 7 the group Γ is a group of type (FL). Therefore, the functors $H^n(\Gamma, \bullet)$ commute with direct limits by [Se], Proposition 4. It follows that $H^n(\Gamma, M) = 0$ for $n > 6g - 9$ and every free Γ -module M . This proves Claim 2. \square

It remains to note that the cohomology dimension of Γ is finite (see Section 1) and apply Lemma 8. This completes the proof of Lemma 9. \blacksquare

Deduction of Theorem 1 from Theorem 2. Now we can prove that Theorem 2 implies Theorem 1. By a result of W. Harvey [5], the boundary $\partial\bar{T}_g \neq \emptyset$. By another result of W. Harvey [5], $\partial\bar{T}_g$ is homotopy equivalent to $\mathcal{C}(X_g)$. By combining Lemma 9 with these results of W. Harvey, we see that Theorem 2 implies the first part of Theorem 1, namely, that $\text{vcd}(\text{Mod}_g) \leq 6g - 9$ if $g \geq 2$.

It remains to prove that Theorem 2 implies the part of Theorem 1 concerned with Mod_2 . First, note that Theorem 2 together with Lemma 9 imply that

$$(1) \quad \text{vcd}(\text{Mod}_2) \leq 6 \cdot 2 - 6 = 3.$$

On the other hand, by Section 1

$$(2) \quad 3g - 3 \leq \text{vcd}(\text{Mod}_g).$$

for all g . By applying (2) to $g = 2$, we see that $3 \leq \text{vcd}(\text{Mod}_2)$. By taking (1) into account, we see that $\text{vcd}(\text{Mod}_2) = 3$. It remains to prove that Mod_2 is virtually a duality group.

10. Theorem *Mod_2 is virtually a duality group.*

Proof. Since $\dim \mathcal{C}(X_2) \leq 2$ and $\mathcal{C}(X_2)$ is simply-connected by Theorem 2, $\mathcal{C}(X_2)$ is homotopy equivalent to a wedge of 2-spheres. Hence $\partial\bar{T}_2$ is also homotopy equivalent to a wedge of 2-spheres. In particular, $H_i(\partial\bar{T}_2) = 0$ if $i \neq 2$, and $H_2(\partial\bar{T}_2)$ is torsion free. Let Γ be a subgroup Mod_2 acting freely on \bar{T}_2 and having finite index in Mod_2 . Replacing, if necessary, Γ by a subgroup of index 2 in Γ , we can assume that the manifold \bar{T}_2/Γ is orientable. It remains to apply Theorem 3 to Γ and $\bar{X} = \bar{T}_2$. \blacksquare

4. The complex of curves and the Hatcher-Thurston complex

Simplicial complexes. By a *simplicial complex* \mathcal{V} we understand a simplicial complex in the sense of E. Spanier [Sp], i.e. a pair consisting of a set V together with a collection of *finite subsets of* V . As usual, we think of \mathcal{V} as a structure on the set V , namely a structure of a simplicial complex. Elements of V are called *the vertices of* \mathcal{V} , and subsets of V from the given collection are called *the simplices of* \mathcal{V} . These data are required to satisfy only one condition: *a subset of a simplex is also a simplex*. The *dimension* of simplex S is defined as $\dim S = (\text{card } S) - 1$. The *dimension* of simplicial complex \mathcal{V} is defined as the maximum of dimensions of its simplices, if such maximum exists, and as the infinity ∞ otherwise.

Geometric realizations. Every simplicial complex \mathcal{V} canonically defines a topological space, which is called the *geometric realization* of \mathcal{V} and denoted by $|\mathcal{V}|$. The idea is to take a copy Δ_S of the standard geometric simplex $\Delta_{\dim S}$ for every simplex S of \mathcal{V} , and to glue simplices Δ_S together in such a way that Δ_T will be a face of Δ_S if $T \subset S$, i.e. if T is a *face* of S in the sense of theory of simplicial complexes. We omit the details.

When we speak about topological properties of simplicial complexes, they should be understood as properties of the geometric realization. The main properties of interest for us, namely, the connectedness and the simply-connectedness, can be defined pure combinatorially in terms of simplicial complexes, but such an approach is cumbersome and hides the main ideas.

Barycentric subdivisions. Every simplicial complex \mathcal{V} canonically defines another simplicial complex, which is called the *barycentric subdivision* of \mathcal{V} and denoted by \mathcal{V}' . The vertices of \mathcal{V}' are the simplices of \mathcal{V} . A set of vertices of \mathcal{V}' , i.e. a set of simplices of \mathcal{V} , is a simplex of \mathcal{V}' if and only if it has the form $\{S_1, S_2, \dots, S_n\}$ for some chain $S_1 \subset S_2 \subset \dots \subset S_n$ of simplices of \mathcal{V} . A vertex v of \mathcal{V} is usually identified with the 0-dimensional simplex $\{v\}$ of \mathcal{V} , and, hence, with a vertex of \mathcal{V}' .

As is well known, taking the barycentric subdivision does not change the geometric realization. In other terms, for any simplicial complex \mathcal{V} there is a canonical homeomorphism between $|\mathcal{V}'|$ and $|\mathcal{V}|$.

Circles on surfaces and their isotopy classes. As usual, we call by a *simple closed curve* on a surface X (not necessarily closed) a one-dimensional closed connected submanifold of X . A simple closed curve on a surface X is also called a *circle* on X . For a circle C in a surface X we will denote *the isotopy class of* C in X by $\langle C \rangle$. The surface X is usually clear from the context, even if C is also a circle in some other relevant surfaces (for example, some subsurfaces of X). For a collection C_1, C_2, \dots, C_n we will denote

by $\langle C_1, C_2, \dots, C_n \rangle$ the set of the isotopy classes $\langle C_i \rangle$ of circles C_i with $1 \leq i \leq n$. In other terms,

$$\langle C_1, C_2, \dots, C_n \rangle = \{ \langle C_1 \rangle, \langle C_2 \rangle, \dots, \langle C_n \rangle \}$$

Recall that a circle on X is called *non-trivial* if it cannot be deformed in X into a point or into a boundary component of X .

Complexes of curves. If X is a compact surface, possibly with non-empty boundary, then *complex of curves* $\mathcal{C}(X)$ is a simplicial complex in the above sense. The vertices of $\mathcal{C}(X)$ are the isotopy classes $\langle C \rangle$ of non-trivial circles C in X . A collection of such isotopy classes is a simplex if and only if it is either empty, or the isotopy classes from this collection can be represented by pair-wise disjoint circles. In other words, if C_1, C_2, \dots, C_n are pair-wise disjoint circles on X , then the set $\langle C_1, C_2, \dots, C_n \rangle$ is a simplex of $\mathcal{C}(X)$, and there are no other simplices (the empty set is the only simplex with $n=0$; its dimension is $n-1=0-1=-1$).

It is well known that if X is a closed orientable surface of genus g and C_1, C_2, \dots, C_n are pair-wise disjoint and pair-wise non-isotopic circles on X , then $n \leq 3g-3$, and there are such collections with $n=3g-3$ (for example, the collection of circles on Fig. 1). It follows that $\dim \mathcal{C}(X) = 3g-4$ if X is a closed orientable surface of genus g .

Hatcher-Thurston complexes. As before, we denote by X_g a closed orientable surface of genus g . A set $\{C_1, C_2, \dots, C_g\}$ of g circles on X_g is called a *geometric cut system* on X_g if the circles C_1, C_2, \dots, C_g are pair-wise disjoint and the complement $X_g \setminus (C_1 \cup \dots \cup C_g)$ is (homeomorphic to) a $2g$ -punctured sphere. If $\{C_1, C_2, \dots, C_g\}$ is a geometric cut system, then we call the set of the isotopy classes $\langle C_1, C_2, \dots, C_g \rangle$ a *cut system*.

Suppose that $\{C_1, C_2, \dots, C_g\}$ is a geometric cut system on X_g . Suppose that $1 \leq i \leq g$, and that C' be a circle on X_g disjoint from circles C_j with $j \neq i$, and transversely intersecting C_i at exactly 1 point. If we replace C_i by C' in $\{C_1, C_2, \dots, C_g\}$, we get another cut system. A *simple move* is the operation of replacing the geometric cut system

$$\{C_1, \dots, C_i, \dots, C_g\} \text{ by the geometric cut system } \{C_1, \dots, C', \dots, C_g\},$$

and also the corresponding operation of replacing the cut system

$$\langle C_1, \dots, C_i, \dots, C_g \rangle \text{ by the cut system } \{ \langle C_1, \dots, C', \dots, C_g \rangle \}.$$

Usually we will describe a simple move by pictures omitting the unchanging circles.

Some sequences of simple moves are *cycles* in the sense that they begin and end at the same geometric cut system. The three special types of cycles, depicted on Fig. 3, are the key ingredients of the construction of the Hatcher-Thurston complexes. It is assumed

that the circles omitted from the pictures are disjoint from the ones presented, and form cut systems with them.

The Hatcher-Thurston complex $\mathcal{HT}(X_g)$ of X_g is a 2-dimensional cell complex (it is not a simplicial complex) constructed as follows. Every cut system $\langle C_1, C_2, \dots, C_g \rangle$ is a 0-cell of $\mathcal{HT}(X_g)$; there are no other 0-cells. If one 0-cell can be obtained from another by a simple move, then these two 0-cells are connected by a 1-cell corresponding to this move; there are no other 1-cells. Clearly, one geometric cut system can be obtained from another one by no more than one simple move, and even a cut system can be obtained from another one by no more than one simple move. Therefore, two 0-cells of $\mathcal{HT}(X_g)$ are connected by no more than one 1-cell. At this moment we have already a 1-dimensional cell complex consisting of the just described 0-cells and 1-cells. It is denoted by $\mathcal{HT}_1(X_g)$. The Hatcher-Thurston complex $\mathcal{HT}(X_g)$ is obtained from $\mathcal{HT}_1(X_g)$ by attaching 2-cells to $\mathcal{HT}_1(X_g)$ along circles resulting from the three special types of cycles, namely, cycles of types (I), (II), and (III).

The definition of $\mathcal{HT}(X_g)$ was suggested by A. Hatcher and W. Thurston [6], who also proved the following fundamental result.

11. Theorem. *If the genus g of X_g is ≥ 2 , then the cell complex $\mathcal{HT}(X_g)$ is connected and simply-connected.*

Proof of Theorem 2. It is based on a construction of a map $\mathcal{J}: \mathcal{HT}_1(X_g) \rightarrow |\mathcal{C}(X_g)|$ such that the following two lemmas hold.

12. Lemma. *If $g \geq 2$, then map \mathcal{J} can be extended to a map $\mathcal{HT}(X_g) \rightarrow |\mathcal{C}(X_g)|$.*

13. Lemma. *If $g \geq 2$, then every loop in $|\mathcal{C}(X_g)|$ is freely homotopic to a loop of the form $\mathcal{J}(\beta)$, where β is a loop in $\mathcal{HT}_1(X_g)$.*

Since $\mathcal{HT}(X_g)$ is simply connected by Theorem 11, Lemmas 12 and 13 together imply that $\mathcal{C}(X_g)$ is simply connected. Therefore, the proof of Theorem 2 is completed modulo Lemmas 12 and 13 and the construction of \mathcal{J} .

In the rest of this section we assume that $g \geq 2$.

The construction of \mathcal{J} . The cell complex $\mathcal{HT}_1(X_g)$ is the geometric realization of a simplicial complex $\mathcal{R}(X_g)$ defined as follows. The set of vertices of $\mathcal{R}(X_g)$ is equal to the set of 0-cells of $\mathcal{HT}(X_g)$. In other words, the vertices of $\mathcal{R}(X_g)$ are the cut systems on X_g . If two 0-cells V_1, V_2 of $\mathcal{HT}(X_g)$ are connected by a 1-cell of $\mathcal{HT}(X_g)$ (i.e. if they are related by a simple move), then the pair $\{V_1, V_2\}$ is a simplex of $\mathcal{R}(X_g)$ (of dimension 1). There are no other simplices; in particular, there are no simplices of dimension ≥ 2 .

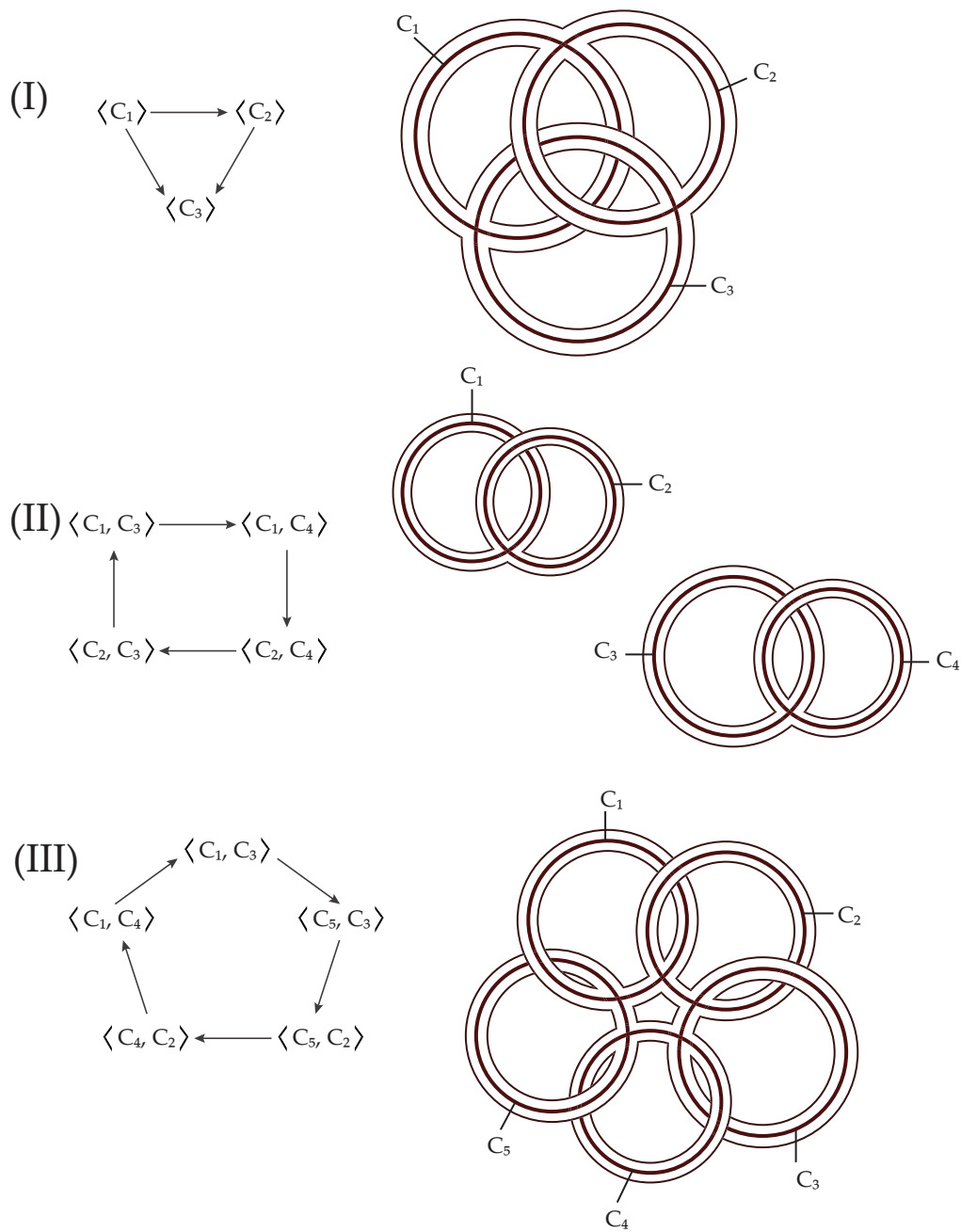


Figure 2: The Hatcher-Thurston moves.

Let $\mathcal{R} = \mathcal{R}(X_g)$, $\mathcal{C} = \mathcal{C}(X_g)$. Let \mathcal{R}' , \mathcal{C}' be the barycentric subdivisions of \mathcal{R} , \mathcal{C} respectively. Let us construct a morphism of simplicial complexes $J: \mathcal{R}' \rightarrow \mathcal{C}'$. Recall that a *morphism of a simplicial complexes* $\mathcal{A} \rightarrow \mathcal{B}$ is defined as a map of the set of vertices of \mathcal{A} to the set of vertices of \mathcal{B} such that the image of a simplex is also a simplex. If $Z = \langle C_1, C_2, \dots, C_g \rangle$ is a vertex of \mathcal{R} considered as a vertex of \mathcal{R}' , we set

$$J(Z) = \langle C_1, C_2, \dots, C_g \rangle,$$

where the right hand side is a simplex of \mathcal{C} considered as a vertex of \mathcal{C}' . If $Z = \{V_1, V_2\}$ is the vertex of \mathcal{R}' corresponding to the edge of \mathcal{R} connecting the vertices V_1 and V_2 of \mathcal{R} , then these two vertices connected by a simple move. Let

$$V_1 = \langle C_1, \dots, C_i, \dots, C_g \rangle \mapsto \langle C_1, \dots, C'_i, \dots, C_g \rangle = V_2$$

be this simple move. Then we set

$$J(Z) = \langle C_1, \dots, C_{i-1}, C_{i+1}, \dots, C_g \rangle$$

where, again, the right hand side is a simplex of \mathcal{C} considered as a vertex of \mathcal{C}' . Obviously, the map J from the set of vertices of \mathcal{R}' to the set of vertices of \mathcal{C}' is a morphism of simplicial complexes $\mathcal{R}' \rightarrow \mathcal{C}'$.

Recall that a morphism of simplicial complexes $f: \mathcal{A} \rightarrow \mathcal{B}$ canonically defines a continuous map $|f|: |\mathcal{A}| \rightarrow |\mathcal{B}|$, called the *geometric realization* of f . Therefore, J leads to a continuous map $|J|: |\mathcal{R}'| \rightarrow |\mathcal{C}'|$. Since the geometric realization of the barycentric subdivision of a complex is canonically homeomorphic to the geometric realization of the complex itself, we may consider $|J|$ as a map $|\mathcal{R}| \rightarrow |\mathcal{C}|$.

Recall that $\mathcal{HT}_1(X_g)$ is the geometric realization of $\mathcal{R} = \mathcal{R}(X_g)$. Therefore, we may define $\mathcal{J}: \mathcal{HT}_1(X_g) \rightarrow |\mathcal{C}(X_g)|$ to be the map $|J|$ considered as map $\mathcal{HT}_1(X_g) \rightarrow |\mathcal{C}(X_g)|$.

14. Proof of Lemma 12. It is sufficient to prove that \mathcal{J} maps every cycle of type (I), (II), or (III) to a loop contractible in \mathcal{C} . We will consider these three types of cycles separately.

Cycles of type (I). Let $\langle C_1, \dots, C_g \rangle$ be a 0-cell of $\mathcal{HT}(X_g)$ involved into a cycle of type (I). Then all circles C_1, \dots, C_g except one remain unchanged under 3 simple moves forming this cycle. We may assume that the circle C_1 is not changing. Then J maps every vertex of \mathcal{R}' which belongs to the geometric realization of this cycle into a vertex of \mathcal{C}' of the form $\langle C_1, \dots \rangle$. Therefore J maps every such vertex into a vertex of \mathcal{C}' contained in the star of the vertex $\langle C_1 \rangle$ of \mathcal{C} , considered as a vertex of \mathcal{C}' (by a standard abuse of notations, we identify $\langle C_1 \rangle$ with $\{\langle C_1 \rangle\}$). Indeed, $\{\langle C_1 \rangle, \langle C_1, \dots \rangle\}$ is an edge of \mathcal{C}' connecting $\{\langle C_1 \rangle\}$ with $\{\langle C_1 \rangle, \langle C_1, \dots \rangle\}$. It follows that the image of this cycle under the morphism J is contained in the star of $\langle C_1 \rangle$ and hence the image of the circle resulting from this cycle under \mathcal{J} is contained in the geometric realization of this star, and hence is contractible in this geometric realization. Therefore it is contractible in $|\mathcal{C}'| = |\mathcal{C}| = |\mathcal{C}(X_g)|$. This completes the proof for the cycles of type (I). \square

Cycles of type (II). Let us consider a cycle of type (II). The simple moves of such a cycle change two circles, and the other $g-2$ circles do not change. Therefore, if $g \geq 3$, then at least one circle of the cut systems from this cycle remains in place under all 4 simple moves of this cycle. This allows to complete the proof in this case in exactly the same way as we dealt with the cycles of type (I).

It remains to consider the case of $g = 2$. In this case each cut system consists of 2 circles and only four circles C_1, C_2, C_3, C_4 are involved in the cycle. See Fig. 2 (II). In this case there exist a non-trivial circle C_0 on X disjoint from C_1, C_2, C_3, C_4 . For example, the union $C_1 \cup C_2$ is contained in a subsurface of X_g diffeomorphic to a torus with one hole. We can take as C_0 the boundary circle of this torus with one hole. Alternatively, we can define C_0 as the circle dividing X_g into two tori with 1 boundary component each such that $C_1 \cup C_2$ is contained in one of them, and $C_3 \cup C_4$ is contained in the other one. (This more symmetric description of C_0 easily implies that C_0 is unique up to isotopy, but we will not need this fact.)

Every 0-cell of $\mathcal{HT}(X_g)$ occurring in our cycle has the form $\langle C_i, C_j \rangle$, where $i = 1$ or 2 and $j = 3$ or 4. In order to describe this cycle in more details, it is convenient to introduce an involution σ on the set $\{1, 2, 3, 4\}$. Namely, we set

$$\sigma(1) = 2, \quad \sigma(2) = 1, \quad \sigma(3) = 4, \quad \sigma(4) = 3.$$

Then every 1-cell contained in our cycle corresponds to a simple move of the form

$$\langle C_i, C_j \rangle \mapsto \langle C_{\sigma(i)}, C_j \rangle, \quad \text{or of the form} \quad \langle C_i, C_j \rangle \mapsto \langle C_i, C_{\sigma(j)} \rangle,$$

where $i = 1$ or 2, and $j = 3$ or 4. In the barycentric subdivision \mathcal{R}' the edge connecting $\langle C_i, C_j \rangle$ with $\langle C_{\sigma(i)}, C_j \rangle$ is subdivided into two edges, connecting the vertex

$$\{ \langle C_i, C_j \rangle, \langle C_{\sigma(i)}, C_j \rangle \}$$

of \mathcal{R}' with the vertices $\langle C_i, C_j \rangle$ and $\langle C_{\sigma(i)}, C_j \rangle$ respectively. Since C_0 is disjoint from the circles C_1, C_2, C_3, C_4 , the images of both these edges under the map J are contained in the star of the vertex $\langle C_0 \rangle$ (more precisely, $\{\langle C_0 \rangle\}$) of \mathcal{C}' . The same argument applies to all edges into which our cycle is subdivided in \mathcal{R}' . It follows that J maps the subdivided cycle into the star of $\langle C_0 \rangle$ in \mathcal{C}' , and hence the geometric realization $\mathcal{J} = |J|$ maps the geometric realization of our cycle into the geometric realization of this star. Therefore, this image is contractible in the geometric realization of this star, and hence in $|\mathcal{C}'| = |\mathcal{C}| = |\mathcal{C}(X_g)|$. This completes the proof for the cycles of type (II). \square

Cycles of type (III). This is the most difficult case. If $g \geq 3$, then one of the circles is not changed under all five moves of the cycle and we can use the same argument as we used for the cycles of type (I) and for the cycles of type (II) in the case $g \geq 3$.

It remains to consider the case of $g = 2$. In this case each cut system consists of 2 circles and only five circles C_1, C_2, C_3, C_4, C_5 are involved in the cycle. See Fig. 2 (III). Let C_0

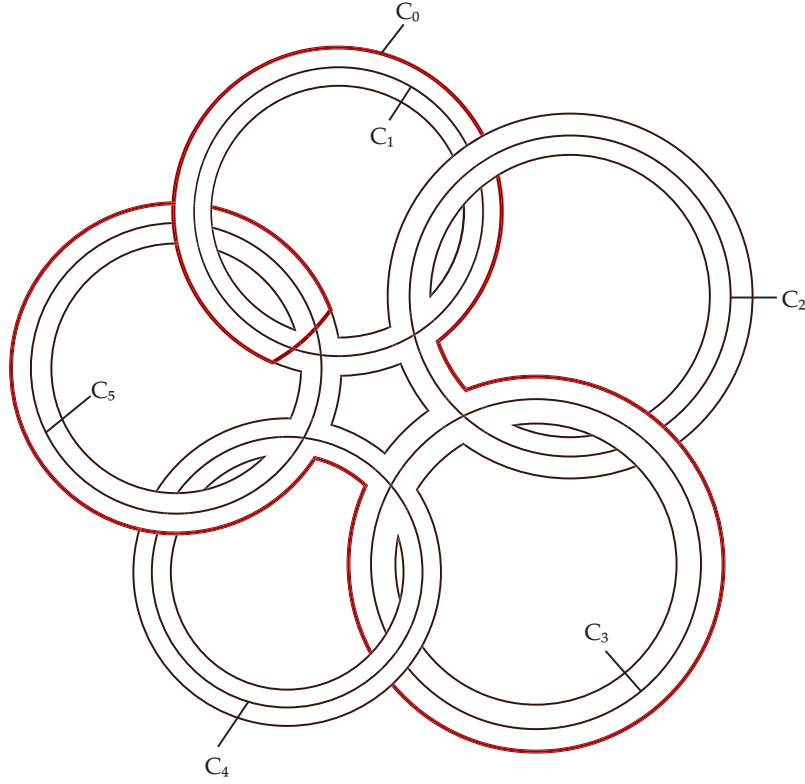


Figure 3: Auxiliary circle for move (III).

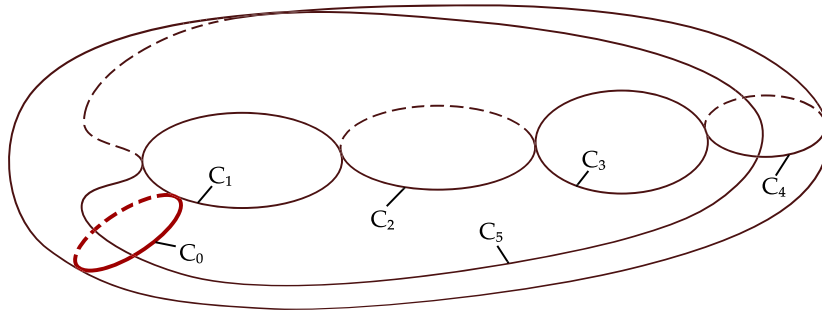


Figure 4: Another view of the auxiliary circle for move (III).

be a circle on X_g disjoint from C_2, C_3, C_4 and intersection each of the circles C_1 and C_5 transversely at one point. One can take as C_0 the circle C_0 on the Fig. 3.

An alternative way to draw such a circle is presented on the Fig. 4. We leave to the interested readers to show that the circles C_0 on these two pictures are isotopic; we will not use this fact.

Let us consider the image under \mathcal{J} of the circle in $\mathcal{HT}_1(X_{\mathfrak{g}})$ resulting from our cycle. This image is the geometric realization of the (simplicial) loop in \mathcal{C}' shown on Fig. 5.

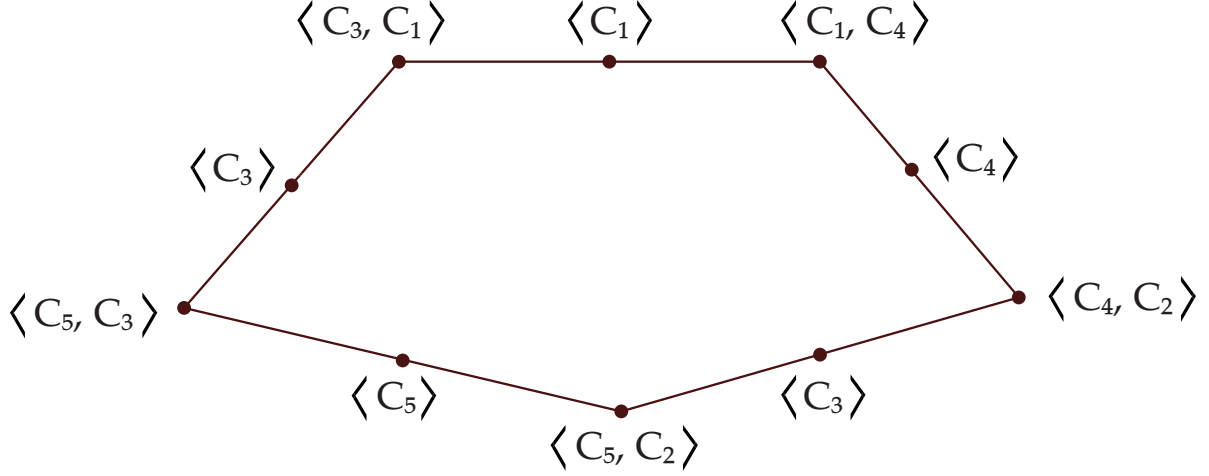


Figure 5: The pentagon.

The subgraph (i.e. a 1-dimensional simplicial subcomplex) of \mathcal{C}' , shown on Fig. 6 contains the above simplicial loop as a subgraph.

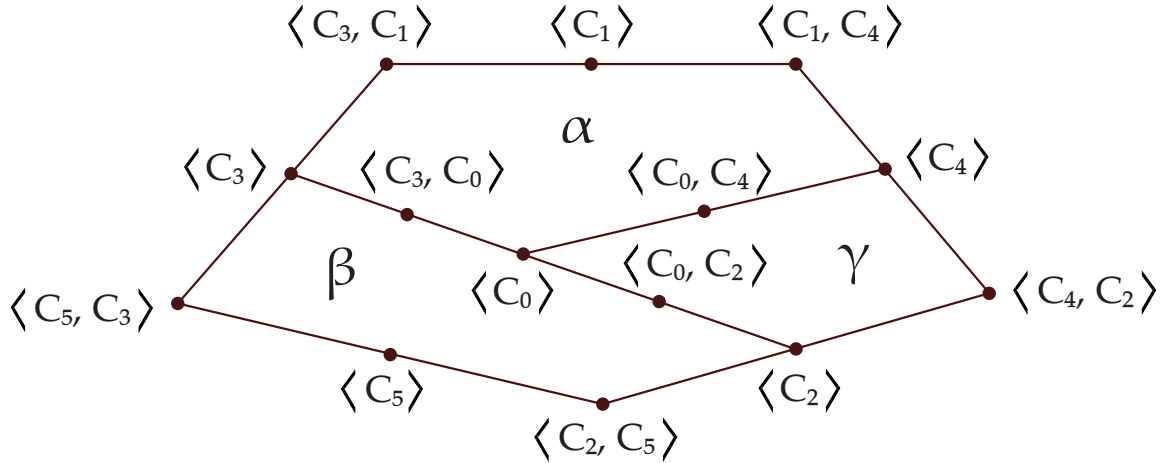


Figure 6: Filling in the pentagon.

The subgraphs bounding the domains α and β on this picture are equal to the images under the map J of the barycentric subdivisions of the two cycles of type (II) in \mathcal{R} shown on Fig. 7. Therefore, their geometric realizations are contractible in $|\mathcal{C}| = |\mathcal{C}(X_g)|$.

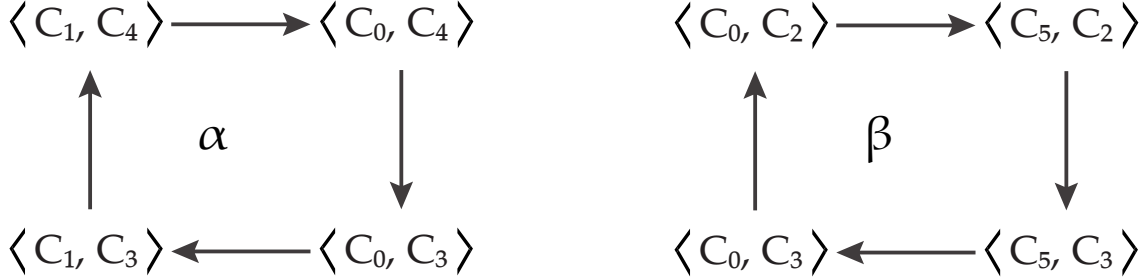


Figure 7: Cycles for α and β .

The subgraph bounding the domain γ on this picture is equal to the image under the map J of the barycentric subdivision of the boundary of the triangle (i.e. a 2-dimensional simplex) $\langle C_0, C_2, C_4 \rangle$ in \mathcal{C} . Therefore, its geometric realization is contractible in $|\mathcal{C}|$.

It follows that the geometric realizations of these 3 loops (subgraphs) are contractible in $\mathcal{HT}(X_g)$. Therefore, the geometric realization of the loop on Fig. 5 is also contractible in $|\mathcal{C}|$. Since this geometric realization is the image of our cycle of type (III), this completes the proof for cycles of type (III), and hence the proof of the lemma. \square \blacksquare

15. Proof of Lemma 13. Every loop in $|\mathcal{C}| = |\mathcal{C}(X_g)|$ is freely homotopic to the geometric realization of a simplicial loop in the 1-skeleton of \mathcal{C} . A simplicial loop in the 1-skeleton of \mathcal{C} is just a sequence

$$(3) \quad \langle C_1 \rangle, \langle C_2 \rangle, \dots, \langle C_n \rangle$$

of vertices of \mathcal{C} such that $\langle C_i \rangle$ is connected by an edge of \mathcal{C} with $\langle C_{i+1} \rangle$ for all $i = 1, 2, \dots, n-1$ and $\langle C_n \rangle$ is connected by an edge with $\langle C_1 \rangle$.

From now on we will interpret $n+1$ as 1.

Without loss of generality we may assume that $\langle C_i \rangle \neq \langle C_{i+1} \rangle$ for every $i = 1, 2, \dots, n$.

CLAIM 1. *Without loss of generality, we can assume that circles C_i are non-separating.*

Proof of the claim. Suppose that C_i is a separating circle. Let Y_0 and Y_1 be two subsurfaces of X_g into which C_i divides X_g . Since X_g is a closed surface, both Y_0 and Y_1

are surfaces with one boundary component resulting from C_i . Since C_i is a non-trivial circle, neither Y_0 , nor Y_1 is a disc. Hence each of surfaces Y_0 and Y_1 has genus ≥ 2 .

Let us first consider the case when both circles C_{i-1} and C_{i+1} are non-separating.

If C_{i-1} and C_{i+1} are contained in the same part Y_j of X_g (where $j=0$ or 1), then we can choose a non-separating circle C'_i in the other part Y_{1-j} of X_g , because Y_{1-j} is a surface of genus ≥ 2 . Then both

$$\langle C_{i-1}, C_i, C_{i+1} \rangle \quad \text{and} \quad \langle C_{i-1}, C'_i, C_{i+1} \rangle$$

are simplices (triangles). Therefore, our loop is homotopic to the loop resulting from replacing $\langle C_i \rangle$ by $\langle C'_i \rangle$ in it. Since the circle C'_i is non-separating in Y_{1-j} , it is non-separating in X_g , and hence our new simplicial loop involves one separating circle less than the original one.

If C_{i-1} and C_{i+1} are contained in different parts of X_g , then $C_{i-1} \cap C_{i+1} = \emptyset$, and hence $\langle C_{i-1} \rangle$ and $\langle C_{i+1} \rangle$ are connected by an edge in \mathcal{C} . Moreover, $\langle C_{i-1}, C_i, C_{i+1} \rangle$ is a simplex (triangle) of \mathcal{C} . It follows that if we delete $\langle C_i \rangle$ from our loop, we get a new loop which is homotopic to the original one. As before, the new loop involves one separating circle less than the original one.

Let us now consider the case when the circle C_{i+1} is separating (and C_i is also separating, as before). We may assume that the circles C_i and C_{i+1} are disjoint (replacing them by isotopic circles, if necessary). Then C_i and C_{i+1} together divide X_g into three parts Z_0, Z_1, Z_2 . Since the circles C_i and C_{i+1} are non-isotopic (by our assumption) and are both non-trivial, each of the surfaces Z_0, Z_1, Z_2 has genus ≥ 1 . Since the circle C_{i-1} is disjoint from C_i , the circle C_{i-1} may intersect no more than two of surfaces Z_0, Z_1, Z_2 . Let Z_k be a part disjoint from C_{i-1} . Let C'_i be some non-separating circle in Z_k (such a circle exists because the genus of Z_k is ≥ 1). Then the circles C_{i-1}, C'_i, C_{i+1} are pair-wise disjoint, and hence both

$$\langle C_{i-1}, C_i, C_{i+1} \rangle \quad \text{and} \quad \langle C_{i-1}, C'_i, C_{i+1} \rangle$$

are simplices (triangles). It follows that if we replace in our loop the vertex $\langle C_i \rangle$ by the vertex $\langle C'_i \rangle$, we will get a new loop homotopic to the original one. Since C'_i is non-separating circle in Z_k , and hence is a non-separating circle in X_g , the new loop involves one separating circle less than the original one.

Finally, in the case when the circle C_{i-1} is separating, the same arguments as in the case when C_{i+1} apply. This allows us replace our loop by a homotopic new loop involving one separating circle less than the original one in this case also.

By repeating the above procedure until there will be no separating circles involved, we can construct a new loop homotopic to the original loop and involving no separating circle. This proves our claim. \square

CLAIM 2. *Without loss of generality, we can assume that, in addition to circles C_i being non-separating, every edge $\langle C_i, C_{i+1} \rangle = \{\langle C_i \rangle, \langle C_{i+1} \rangle\}$ can be completed to a cut system.*

Proof of the claim. By Claim 1, we can assume that all circles C_i are non-separating. Suppose that $\langle C_i, C_{i+1} \rangle$ cannot be completed to a cut system. Since $\langle C_i \rangle$ and $\langle C_{i+1} \rangle$ are connected by an edge of \mathcal{C} , we may assume that the circle C_i and C_{i+1} are disjoint. Then $\langle C_i, C_{i+1} \rangle$ cannot be completed to a cut system only if the union $C_i \cup C_{i+1}$ divides our surface X_g into two parts (it cannot divide X_g into three parts because neither C_i , nor C_{i+1} divide X_g). Let these two parts be Y_0 and Y_1 , so that $Y_0 \cup Y_1 = X_g$ and $Y_0 \cap Y_1 = C_i \cup C_{i+1}$. Since $\langle C_i \rangle$ and $\langle C_{i+1} \rangle$ are assumed to be different, and hence C_i and C_{i+1} are not isotopic, each of the subsurfaces Y_0 and Y_1 has genus ≥ 1 .

Let us choose some circle C'_i contained in Y_0 and non-separating in Y_0 (this is possible because the genus of Y_0 is ≥ 1). Then C'_i is non-separating in X_g also and the circles C_i, C'_i, C_{i+1} are pair-wise disjoint. In particular, $\langle C_i, C'_i, C_{i+1} \rangle$ is 2-simplex (triangle) of \mathcal{C} . Moreover, since C'_i is non-separating in Y_0 , it is also non-separating in both $X_g \setminus C_i$ and $X_g \setminus C_{i+1}$. Therefore both unions $C_i \cup C'_i$ and $C'_i \cup C_{i+1}$ do not divide X_g into two parts. It follows that both pairs $\langle C_i, C'_i \rangle$ and $\langle C'_i, C_{i+1} \rangle$ can be completed to cut systems.

Let us replace the edge connecting $\langle C_i \rangle$ with $\langle C_{i+1} \rangle$ in our loop by the following two edges: the first one connecting $\langle C_i \rangle$ with $\langle C'_i \rangle$; the second one connecting $\langle C'_i \rangle$ with $\langle C_{i+1} \rangle$. Since $\langle C_i, C'_i, C_{i+1} \rangle$ is 2-simplex, the new loop is homotopic to the original one. Since both pairs $\langle C_i \cup C'_i \rangle$ and $\langle C'_i \cup C_{i+1} \rangle$ can be completed to cut systems, the new loop has less edges which cannot be completed to cut systems than the original one.

By repeating this procedure we can construct a new loop homotopic to the original one and having the required properties. This completes the proof of the claim. \square

By Claim 2, it is sufficient to consider loops (3) in \mathcal{C} such that every C_i is a non-separating circle, and every pair $\langle C_i, C_{i+1} \rangle$ can be extended to a cut system. Given such a loop (3), we consider the following loop in the barycentric subdivision \mathcal{C}'

$$(4) \quad \langle C_1 \rangle, \langle C_1, C_2 \rangle, \langle C_2 \rangle, \langle C_2, C_3 \rangle, \dots, \langle C_{n-1}, C_n \rangle, \langle C_n \rangle, \langle C_n, C_1 \rangle$$

For every $i = 1, 2, \dots, n$ there is an edge of this loop connecting $\langle C_i \rangle$ with $\langle C_i, C_{i+1} \rangle$, and an edge connecting $\langle C_i, C_{i+1} \rangle$ with $\langle C_{i+1} \rangle$ (recall that $n+1$ is interpreted as 1); there are no other edges. Clearly, the loops (3) and (4) have the same geometric realization.

Let us complete each pair $\langle C_i, C_{i+1} \rangle$ to a cut system Z_i . Clearly, Z_i has the form $Z_i = \langle C_i, C_{i+1}, C_3^i, \dots, C_g^i \rangle$ if $g \geq 3$, and $Z_i = \langle C_i, C_{i+1} \rangle$ if $g = 2$.

Let us temporarily fix an integer i between 1 and n . Let us cut our surface X_g along C_i and denote the result by X_i^0 . The surface X_i^0 has two boundary components and

its genus is equal to $g-1$. Next, let us glue two discs to the components of ∂X_i^0 and denote the result by X_i^1 . Clearly, X_i^1 is a closed surface of genus $g-1$. We may assume that $\{C_i, C_{i+1}, C_3^i, \dots, C_g^i\}$ and $\{C_{i-1}, C_i, C_3^{i-1}, \dots, C_g^{i-1}\}$ are geometric cut systems on X_g . Then $\{C_{i+1}, C_3^i, \dots, C_g^i\}$ and $\{C_{i-1}, C_3^{i-1}, \dots, C_g^{i-1}\}$ are geometric cut systems on X_i^1 (because all circles involved are contained in $X_i^0 \subset X_i^1$). Therefore, by taking the isotopy classes in X_i^1 instead of X_g , we can define two cut system on X_i^1 as follows: $Z_i^0 = \langle C_{i+1}, C_3^i, \dots, C_g^i \rangle$ and $Z_i^1 = \langle C_{i-1}, C_3^{i-1}, \dots, C_g^{i-1} \rangle$. Because $\mathcal{HT}(X_i^1)$ is connected, Z_i^0 can be joined with Z_i^1 by a path in $\mathcal{HT}(X_i^1)$, and hence by a path in the 1-skeleton of $\mathcal{HT}(X_i^1)$. It follows that Z_i^0 can be joined with Z_i^1 by a path in the simplicial complex $\mathcal{R}'(X_i^1)$. Let us denote this path by α_i . Let $\langle D_1, D_2, \dots, D_n \rangle$ be a vertex of $J(\alpha_i)$ (where $n = g-1$ or $g-2$ by the construction of J). Because $X_i^1 \setminus \text{int } X_i^0$ is a union of two disjoint discs we may assume, replacing the circles D_1, D_2, \dots, D_n by circles isotopic to them in X_i^1 , if necessary, that $D_1, D_2, \dots, D_n \subset \text{int } X_i^0$. Then $\langle C_i, D_1, D_2, \dots, D_n \rangle$ is a vertex of $\mathcal{C}'(X_g)$. By adding in this way $\langle C_i \rangle$ to all vertices of the path α_i , we will obtain a sequence of vertices of $\mathcal{C}' = \mathcal{C}'(X_g)$. Clearly, this sequence is a simplicial path in \mathcal{C}' , and, moreover, it is equal to $J(\beta_i)$ for some simplicial path β_i in $\mathcal{R}'(X_g)$.

Now, let us put together all paths β_i for $i = 1, 2, \dots, n$. Let β be the resulting loop. In order to complete the proof, it is sufficient to show that the geometric realization of the loop (4) is freely homotopic to $J(\beta)$. In order to prove this, it is sufficient, in turn, to prove that for every i the path $J(\beta_i)$, which connects Z_{i-1} with Z_i , is homotopic relatively to the endpoints to the path

$$\langle Z_{i-1} \rangle, \langle C_{i-1}, C_i \rangle, \langle C_i \rangle, \langle C_i, C_{i-1} \rangle, \langle Z_i \rangle.$$

But both these paths are contained in the star of $\langle C_i \rangle$ in \mathcal{C}' . Therefore they are homotopic relatively to the endpoints. This completes the proof of the lemma. ■

Lemmas 13 and 12 are now proved. As we saw, these lemmas imply Theorem 2. In addition, Theorem 1 follows from Theorem 2 by the results of Section 3. Therefore, our main theorems, namely, Theorems 1 and 2 are proved.

5. Beyond the simply-connectedness of $\mathcal{C}(X_g)$

The connectedness and simply-connectedness of $\mathcal{C}(X_g)$. The connectedness of $\mathcal{C}(X_g)$ can be proved by a direct argument, which we leave as an exercise to an interested reader.

A natural approach to proving the simply-connectedness of $\mathcal{C}(X_g)$ is to look for a reduction of this problem to the simply-connectedness of $\mathcal{HT}(X_g)$. The latter was proved

by A. Hatcher and W. Thurston [6]. The complexes $\mathcal{C}(X_g)$ and $\mathcal{HT}(X_g)$ are not related in any direct and obvious manner. Still, it is possible to relate them in a not quite direct (but canonical) way and deduce the simply-connectedness and connectedness of $\mathcal{C}(X_g)$ from the corresponding properties of $\mathcal{HT}(X_g)$. This deduction is the heart of the paper [Iv2] and is presented in Section 4 above. This deduction allows to prove that $\mathcal{C}(X_g)$ is simply-connected if $g \geq 2$ (note that $\mathcal{C}(X_g)$ is not even connected if $g = 1$).

Suppose that $g \geq 2$. In view of the results of Sections 1 and 3, the connectedness of $\mathcal{C}(X_g)$ implies an estimate of $\text{vcd}(\text{Mod}_g)$ better than the trivial estimate $\text{vcd}(\text{Mod}_g) \leq 6g - 7$, and the simply-connectedness implies an even better estimate. Namely, the connectedness of $\mathcal{C}(X_g)$ implies that $\text{vcd}(\text{Mod}_g) \leq 6g - 8$, and the simply-connectedness implies that $\text{vcd}(\text{Mod}_g) \leq 6g - 9$.

The complexes of curves and the Hatcher-Thurston complexes. At the first sight, deducing the simply-connectedness of $\mathcal{C}(X_g)$ from the simply-connectedness of $\mathcal{HT}(X_g)$ seems to be somewhat artificial. This was my opinion in 1983 and for many years to follow. Much later, with the benefit of the hindsight, I started to think that this opinion was short-sighted. In fact, this deduction contains the nuclei of many arguments used later to study the complexes of curves, starting with my papers [Iv3], [Iv4], and [Iv6, Iv7, Iv8]. Nowadays these arguments are among the most natural tools of trade.

There was also a better reason to be unsatisfied with such a deduction. Namely, such a deduction cannot be extended to prove higher connectivity of $\mathcal{C}(X_g)$ when it is expected, since the complex $\mathcal{HT}(X_g)$ is only 2-dimensional. The idea to generalize the whole paper of A. Hatcher and W. Thurston [6] to a higher-dimensional complexes, yet to be constructed, appeared to be too far-fetched. This is the *road not taken*.

A natural alternative to constructing higher-dimensional versions of $\mathcal{HT}(X_g)$, is to try to apply the ideas of [6] to $\mathcal{C}(X_g)$ directly. The main tool of Hatcher and Thurston [6] is the Morse-Cerf theory [Cerf], an analogue of the Morse theory for families of functions with 1 parameter. In order to work with the complex of curves one needs, first of all, to modify the Morse-Cerf theory in such a way that it will lead to result about $\mathcal{C}(X_g)$, and not about $\mathcal{HT}(X_g)$. Also, one needs to at least partially extend the Morse-Cerf theory to the families of functions on surfaces with arbitrary number of parameters. The latter would be necessary even if the high-dimensional versions of $\mathcal{HT}(X_g)$ would be constructed.

The classification of singularities and the Morse-Cerf theory. The Morse theory deals with individual functions, which may be considered as families of functions with 0 parameters. J. Cerf [Cerf] extended the Morse theory to families of functions with 1 parameter. Families with 2 parameters also appear in [Cerf], but they are not arbitrary: they are constructed in order to deform families with 1 parameter. The main difficulty in extending the Morse-Cerf theory to families with an arbitrary number of parameters

results from the lack of classification of singularities of functions in generic families of functions depending on several parameters. The Morse theory requires only the classification of singularities of generic functions. The Cerf theory [Cerf] requires only the classification of singularities of functions in generic families with 1 parameter.

The term *classification* is used here in a precise and very strong sense. A *singular point* of a smooth function $f: M \rightarrow \mathbf{R}$, where M is a smooth manifold, is defined as a point $x \in M$ such that the differential $d_x f$ is equal to 0. Two singular points x, y of functions $f: M \rightarrow \mathbf{R}, g: N \rightarrow \mathbf{R}$ respectively are called *equivalent* if $g \circ \varphi = f + c$ for some real constant c and some diffeomorphism φ between a neighborhood of x in M and a neighborhood of y in N , such that $\varphi(x) = y$. A *singularity* can be defined as an equivalence class of singular points of smooth functions. A *classification of singularities* of function in some class consists of a list of all possible singularities of functions in this class, and explicit formulas for representatives of each singularity in this list. An explicit formula for a representative is called a *normal form* of the corresponding singularity. Usually one takes as a normal form of a singularity a polynomial in several variables with 0 being the singular point in question.

The most important classes of functions for the applications are the classes of functions occurring in *generic* families of functions with a given number m of parameters. The singularities of such functions are called the singularities of *codimension* $\leq m$. The *codimension of a singularity* is defined as the smallest m such that the singularity is of codimension $\leq m$. The codimension of a singularity is the main measure of its complexity from the point of view of applications.

By 1983 V. I. Arnold and his students to a big extent completed Arnold's program of classification of singularities of functions (of course, the nature of Arnold's program is such that it never can be completed). The book [AVG], presenting the main results of Arnold's program, appeared in 1982.

Arnold discovered, in particular, that the codimension is not the best measure of complexity of a singularity for the purposes of classification. Instead, the dimension of the space of deformations of a singularity is a more appropriate characteristic. This dimension, properly defined, is called the *modality* of a singularity. The singularities of the modality 0 are, essentially, the ones which cannot be deformed to a non-equivalent singularity by a small deformation. They are called *simple singularities*. The singularities of modality 1 are called *unimodal*. The simple singularities are simplest to classify, the next case being the unimodal singularities. Arnold classified the simple singularities already in 1972 [A1], [A2] (see also [A4]), and the unimodal ones in 1974 [A3].

As a corollary of the classification of simple singularities, Arnold found a classification all singularities of codimension ≤ 5 (they are all simple). The classification of unimodal singularities lead to a classification of singularities of codimension ≤ 9 (they are all either simple or unimodal). But there is no hope to find a classification (in the above sense) of singularities of arbitrary codimension (or, what is the same, of arbitrary modality).

As it eventually turned out, in the context of our problem the Morse functions are the worst ones, and one can bypass the classification of singularities entirely. This was done in [Iv3]. The next section tells more about the story behind [Iv3].

6. Reminiscences: $\text{vcd}(\text{Mod}_g)$ in Leningrad, 1983

The virtual cohomology dimension $\text{vcd}(\text{Mod}_g)$ and the connectivity of $\mathcal{C}(X_g)$. In the Spring of 1983 I was working, among other things, on the problem of computing the virtual cohomology dimension of Mod_g . The arguments of Sections 1 and 3 were in my mind from the very beginning, despite the fact that I learned the Bieri–Eckmann theory [1] only in the process of working on this problem, and I was only vaguely familiar with the Borel–Serre theory [2]. It seems that these ideas were in the air at the time.

Since I admired the Hatcher–Thurston paper [6] and studied it in details, it was only natural to try to deduce the simply-connectedness of $\mathcal{C}(X_g)$ from the main result of [6], the simply-connectedness of $\mathcal{HT}(X_g)$. This lead to the arguments of Section 4 and a proof of Theorem 2. In turn, this immediately lead to a proof of Theorem 1.

This work was done in March and April of 1983 and presented at Rokhlin’s Topology Seminar in Leningrad in April. In May I prepared a research announcement which included Theorems 1 and 2, as well as other results about Teichmüller modular groups which I proved starting from December of 1982. The announcement was presented by Academician L.D. Faddeev to *Doklady of Academy of Sciences of the USSR* (known also as *DAN*) at May 16, 1983. It was published [Iv1] in the first months of the next year. These results were also included in *Short Communications* distributed at least among the participants of the Warsaw Congress in August of 1983.

The Morse-Cerf theory and the complexes of curves. Eventually it turned out that there is a method to apply an ideal version of the Morse-Cerf theory directly to $\mathcal{C}(X_g)$ without using the Hatcher-Thurston complex $\mathcal{HT}(X_g)$ as an intermediary. I found such a method in Summer of 1983. In fact, it turned out that it is much easier to apply the Morse-Cerf theory directly to $\mathcal{C}(X_g)$ than to the Hatcher-Thurston complex $\mathcal{HT}(X_g)$, not to say about using $\mathcal{HT}(X_g)$ as an intermediary.

As expected, the method allowed in principle to prove that the complex of curves $\mathcal{C}(X_g)$ is n -connected if a classification of singularities up to codimension $n + 1$ is available. The method required that the normal forms of these singularities were not too complicated in a precise sense. The well known normal forms of singularities of codimension ≤ 2 are trivially not too complicated in this sense. This allowed to reprove the connectedness and the simply-connectedness of $\mathcal{C}(X_g)$ without using the Hatcher-Thurston theory. The relation with the classification of singularities was completely parallel to the Hatcher-Thurston theory: in order to prove that $\mathcal{HT}(X_g)$ is connected (respectively,

simply-connected), Hatcher and Thurston used the classification of singularities of codimension ≤ 1 (respectively, of codimension ≤ 2).

Arnold's classification of singularities of codimension ≤ 5 immediately implied that these singularities are simple enough for my method to work. This allowed to prove that $\mathcal{C}(X_g)$ is 3-connected if $g \geq 3$, and is 4-connected if $g \geq 4$. In view of Lemma 6, this implied that $\text{vcd}(\text{Mod}_g) \leq 6g - 11$ if $g \geq 3$, and $\text{vcd}(\text{Mod}_g) \leq 6g - 12$ if $g \geq 4$. After checking the properties of the normal forms of singularities of codimension ≤ 6 , I proved that, moreover, $\mathcal{C}(X_g)$ is 5-connected if $g \geq 4$. In view of Lemma 6, this implied that $\text{vcd}(\text{Mod}_g) \leq 6g - 13$ if $g \geq 4$.

I planned to go further through Arnold's lists of normal forms, and, in particular, to look at all singularities of codimension ≤ 9 . It was clear that such a straightforward approach relying on normal forms will exhaust its potential soon. But the experience with the normal forms at the initial part of Arnold's list lead me to believe that all singularities of high codimension are very simple for the purposes of my method, and that there should be a way to bypass the normal forms and the classification. This work was interrupted by a trip to Warsaw to attend the Warsaw Congress.

Warsaw Congress, August 1983. By the time of the Warsaw Congress I had proved that $\text{vcd}(\text{Mod}_g) \leq 6g - 11$ if $g \geq 3$ and $\text{vcd}(\text{Mod}_g) \leq 6g - 13$ if $g \geq 4$. It was clear that the method does not stop there. I was thrilled when W. Thurston showed up for my short talk at the Congress. Unfortunately, my command of spoken English was negligible, and I spoke in Russian. Volodya Turaev acted as an interpreter. After the talk Thurston suggested to discuss my talk and to tell me the news related to results and problems discussed in my talk. Note that at the time the communication between Western and Soviet mathematicians was anything but easy, and the Warsaw Congress presented a unique opportunity to learn about thing not yet published or even not written down. During this discussion (with Volodya Turaev continuing to serve as an interpreter) Thurston told that J. Harer computed the virtual cohomological dimension of Mod_g . Unfortunately, Thurston forgot the actual value of $\text{vcd}(\text{Mod}_g)$. After being pressed, Thurston agreed that the value of $\text{vcd}(\text{Mod}_g)$ is "as expected".

For me, the value of $\text{vcd}(\text{Mod}_g)$ being "as expected" meant that everything is parallel to the Borel-Serre theory [2]. In particular, $\mathcal{C}(X_g)$ is homotopy equivalent to a bouquet of spheres of dimension equal to the topological dimension of $\mathcal{C}(X_g)$, i.e. to $3g - 4$ (cf. Remark 3.5 in [Iv-J]). If this is the case, then the Bieri-Eckmann theory [1] implies that $\text{vcd}(\text{Mod}_g) = \dim T_g - (3g - 4) - 1$. Since $\dim T_g = 6g - g$, this means that $\text{vcd}(\text{Mod}_g) = (6g - 6) - (3g - 4) - 1 = 3g - 3$. In fact, the Bieri-Eckmann theory [1], together with the simply-connectedness of $\mathcal{C}(X_g)$, implies that $\text{vcd}(\text{Mod}_g) = 3g - 3$ if and only if $\mathcal{C}(X_g)$ is $(3g - 5)$ -connected, but not $(3g - 4)$ -connected (for $g \geq 2$).

My methods were clearly not sufficient to prove that $\mathcal{C}(X_g)$ is $(3g - 5)$ -connected (which

is not surprising, because it is indeed not $(3g-5)$ -connected), and I abandoned the project for a couple of months.

A misunderstanding. I am inclined to think that W. Thurston wasn't at fault when he said that the value of $\text{vcd}(\text{Mod}_g)$ is "*as expected*" and did not remember the correct formula. W. Thurston was thinking about deeper issues than a formula for $\text{vcd}(\text{Mod}_g)$. Most likely, he was thinking about the *reasons allowing to find* the value of the virtual cohomology dimension of Mod_g , and they were "*as expected*". My reasons to expect that $\text{vcd}(\text{Mod}_g) = 3g-3$ were based on an analogy between Teichmüller modular groups and arithmetic groups. As it seems now, I expected this analogy to hold with more details than it actually holds. Since $3g-3$ is equal to the maximal rank of the abelian (and of the solvable) subgroups of Mod_g , the analogy with the arithmetic groups suggested that $3g-3$ should be the answer. While this analogy is a very good guiding principle, it is not complete. Moreover, this lack of completeness makes the theory of Teichmüller modular groups much more interesting than it would be otherwise.

Autumn of 1983. After returning from Warsaw, I wrote to J. Birman, asking, in particular, about what exactly was proved by J. Harer about the order of connectedness of $\mathcal{C}(X_g)$ and the virtual cohomology dimension $\text{vcd}(\text{Mod}_g)$. At that time crossing the USSR border usually took one-two months for a letter. The reply from J. Birman arrived only at the late autumn of 1983. In her reply she wrote me that according to J. Harer $\mathcal{C}(X_g)$ is $(2g-3)$ -connected, but is not $2g-2$ -connected and that $\text{vcd}(\text{Mod}_g) = 4g-5$ for $g \geq 2$.

I immediately realized that this is exactly what my methods can in principle provide. Independently of the form the classification of singularities takes in higher codimension, higher than $(2g-3)$ -connectivity could not be proved by my methods because already Morse functions prevent this. After this I quickly proved that all singularities of higher codimension are indeed simpler than the Morse singularities for the purposes of my method. See [Iv3], Subsection 2.1 and Lemma 2.2 for the key idea. This allowed me to complete the proof of $(2g-3)$ -connectedness of $\mathcal{C}(X_g)$ by the end of 1983.

About the same time the preprint of [Har1] arrived. It contained, in particular, a beautiful combinatorial proof of the fact that $\mathcal{C}(X_g)$ is homotopy equivalent to a $(2g-2)$ -dimensional CW-complex. This result is independent from the main part of [Har1], which is concerned with $(2g-3)$ -connectedness of $\mathcal{C}(X_g)$. Together with the $(2g-3)$ -connectedness of $\mathcal{C}(X_g)$, this result implies that $\text{vcd}(\text{Mod}_g) = 4g-5$ for $g \geq 2$. Combined with my proof of the $(2g-3)$ -connectedness of $\mathcal{C}(X_g)$, this leads to a computation of $\text{vcd}(\text{Mod}_g)$ largely independent from Harer's one.

Harer's exposition was somewhat obscure for my taste, and I found a different version of his proof of homotopy equivalence of $\mathcal{C}(X_g)$ to a $(2g-2)$ -dimensional CW-complex. It brings to the light the fact that the basic properties of the Euler characteristic (never mentioned by Harer) are behind Harer's combinatorial arguments.

All these results and their analogues for non-orientable surfaces were published in [Iv3].

A lemma in Harer's paper. Harer's paper [Har1] contains at least one gap: the proof of Lemma 3.6 is not correct and, I believe, cannot be saved. But I always believed that the lemma is correct and can be proved by other means. Unfortunately, to this day (August 21, 2015) I am not aware of any proof of this lemma. In order to prove a similar result in other situation, namely, Lemma 2.5 in [Iv4], I had to use deep results from the theory of minimal surfaces. It is desirable to find an elementary proof of Lemma 2.5 from [Iv4], as also any, preferably elementary, proof of Harer's Lemma 3.6.

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September 30, 2015

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